Calculation of neutrino cross sections and the `nusigma` neutrino-nucleon scattering Monte Carlo

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1 Theoretical background

We will here write down the cross sections for (anti)neutrino scattering off neutrons and protons. In these expressions, we will use the usual kinematic variables x and y. x is the parton's fraction of the nucleon's momentum and y is energy fraction going into other things than the lepton, i.e.

$$y = \frac{E_\nu - E_{\text{final lepton}}}{E_\nu} = 1 - \frac{E_{\text{final lepton}}}{E_\nu}$$

x and y are related to the momentum transfer (squared) by

$$Q^2 = 2ME_\nu xy$$

The ranges for x and y are $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

All parton distributions ($u(x)$,...) are for protons.

Note that in the expressions below, the charged lepton mass has been neglected. For the $\nu_\tau/\bar{\nu}_\tau$ charged current cross section, the tau mass has been included in our calculations though. We have followed the work in Ref. [5] to modify the expression to allow for a non-zero tau mass in these cases. In all the expressions and figures below, we focus on electron and muon (anti)neutrinos.

1.1 Charged current $\nu - p$ cross section

The charged current neutrino-proton cross section is given by

$$\frac{d^2\sigma}{dxdy} = \frac{2G_F^2 ME_\nu}{\pi} \left(\frac{M_W^2}{Q^2 + M_W^2} \right)^2 x \left\{ [d(x) + s(x) + b(x)] + [\bar{u}(x) + \bar{c}(x)] (1-y)^2 \right\}$$

where M is the nucleon mass.

1.2 Charged current $\bar{\nu} - p$ cross section

The charged current antineutrino-proton cross section is given by

$$\frac{d^2\sigma}{dxdy} = \frac{2G_F^2 M E_\nu}{\pi}\left(\frac{M_W^2}{Q^2+M_W^2}\right)^2 x\left\{[u(x)+c(x)](1-y)^2+\left[\bar{d}(x)+\bar{s}(x)+\bar{b}(x)\right]\right\}$$

1.3 Charged current $\nu - n$ cross section

We assume isospin symmetry, meaning that the down and up quark content of the neutron is the same as in the proton, but with u and d interchanged. This means that we can get the parton distributions for the neutron from those of the proton in the following way,

$$\begin{aligned} u_n(x) &= d_p(x) \equiv d(x) \\ d_n(x) &= u_p(x) \equiv u(x) \\ \bar{u}_n(x) &= \bar{d}_p(x) \equiv \bar{d}(x) \\ \bar{d}_n(x) &= \bar{u}_p(x) \equiv \bar{u}(x) \end{aligned}$$

Remembering that all parton distributions (except where otherwise notet), refer to those of the proton, we can then write the charged current neutrino-neutron cross section as

$$\frac{d^2\sigma}{dxdy} = \frac{2G_F^2 M E_\nu}{\pi}\left(\frac{M_W^2}{Q^2+M_W^2}\right)^2 x\left\{[u(x)+s(x)+b(x)]+\left[\bar{d}(x)+\bar{c}(x)\right](1-y)^2\right\}$$

1.4 Charged current $\bar{\nu} - n$ cross section

With the same isospin symmetry assumption as above, the charged current neutrino-neutron cross section is given by

$$\frac{d^2\sigma}{dxdy} = \frac{2G_F^2 M E_\nu}{\pi}\left(\frac{M_W^2}{Q^2+M_W^2}\right)^2 x\left\{[d(x)+c(x)](1-y)^2+\left[\bar{u}(x)+\bar{s}(x)+\bar{b}(x)\right]\right\}$$

1.5 Neutral current $\nu - p$ cross section

The neutral current neutrino-proton cross section is given by

$$\begin{aligned} \frac{d^2\sigma}{dxdy} &= \frac{G_F^2 M E_\nu}{2\pi}\left(\frac{M_Z^2}{Q^2+M_Z^2}\right)^2 x \\ &\quad \Big[\left\{(g_V+g_A)^2+(g_V-g_A)^2(1-y)^2\right\}[u(x)+c(x)] \end{aligned}$$

$$
\begin{aligned}
&+ \left\{(g'_V + g'_A)^2 + (g'_V - g'_A)^2(1-y)^2\right\}[d(x) + s(x) + b(x)] \\
&+ \left\{(g_V - g_A)^2 + (g_V + g_A)^2(1-y)^2\right\}[\bar{u}(x) + \bar{c}(x)] \\
&+ \left\{(g'_V - g'_A)^2 + (g'_V + g'_A)^2(1-y)^2\right\}\left[\bar{d}(x) + \bar{s}(x) + \bar{b}(x)\right]\Big]
\end{aligned}
$$

where

$$
\begin{aligned}
g_V &= \frac{1}{2} - \frac{4}{3}\sin^2\theta_W \\
g_A &= \frac{1}{2} \\
g'_V &= -\frac{1}{2} + \frac{2}{3}\sin^2\theta_W \\
g'_A &= -\frac{1}{2}
\end{aligned}
$$

1.6 Neutral current $\bar{\nu} - p$ cross section

The neutral current antineutrino-proton cross section is given by

$$
\begin{aligned}
\frac{d^2\sigma}{dxdy} &= \frac{G_F^2 M E_\nu}{2\pi}\left(\frac{M_Z^2}{Q^2 + M_Z^2}\right)^2 x \\
&\Big[\left\{(g_V - g_A)^2 + (g_V + g_A)^2(1-y)^2\right\}[u(x) + c(x)] \\
&+ \left\{(g'_V - g'_A)^2 + (g'_V + g'_A)^2(1-y)^2\right\}[d(x) + s(x) + b(x)] \\
&+ \left\{(g_V + g_A)^2 + (g_V - g_A)^2(1-y)^2\right\}[\bar{u}(x) + \bar{c}(x)] \\
&+ \left\{(g'_V + g'_A)^2 + (g' - V + g'_A)^2(1-y)^2\right\}\left[\bar{d}(x) + \bar{s}(x) + \bar{b}(x)\right]\Big]
\end{aligned}
$$

1.7 Neutral current $\nu - n$ cross section

Again, assuming isospin symmetry, the neutral current neutrino-neutron cross section is given by

$$
\begin{aligned}
\frac{d^2\sigma}{dxdy} &= \frac{G_F^2 M E_\nu}{2\pi}\left(\frac{M_Z^2}{Q^2 + M_Z^2}\right)^2 x \\
&\Big[\left\{(g_V + g_A)^2 + (g_V - g_A)^2(1-y)^2\right\}[d(x) + c(x)] \\
&+ \left\{(g'_V + g'_A)^2 + (g'_V - g'_A)^2(1-y)^2\right\}[u(x) + s(x) + b(x)] \\
&+ \left\{(g_V - g_A)^2 + (g_V + g_A)^2(1-y)^2\right\}\left[\bar{d}(x) + \bar{c}(x)\right]
\end{aligned}
$$

$$+\left\{(g'_V - g'_A)^2 + (g'_V + g'_A)^2(1-y)^2\right\}\left[\bar{u}(x) + \bar{s}(x) + \bar{b}(x)\right]\Big]$$

1.8 Neutral current $\bar{\nu} - n$ cross section

Again, assuming isospin symmetry, the neutral current antineutrino-neutron cross section is given by

$$\begin{aligned}
\frac{d^2\sigma}{dxdy} &= \frac{G_F^2 M E_\nu}{2\pi}\left(\frac{M_Z^2}{Q^2+M_Z^2}\right)^2 x \\
&\Big[\left\{(g_V - g_A)^2 + (g_V + g_A)^2(1-y)^2\right\}[d(x) + c(x)] \\
&+ \left\{(g'_V - g'_A)^2 + (g'_V + g'_A)^2(1-y)^2\right\}[u(x) + s(x) + b(x)] \\
&+ \left\{(g_V + g_A)^2 + (g_V - g_A)^2(1-y)^2\right\}\left[\bar{d}(x) + \bar{c}(x)\right] \\
&+ \left\{(g'_V + g'_A)^2 + (g' - V + g'_A)^2(1-y)^2\right\}\left[\bar{u}(x) + \bar{s}(x) + \bar{b}(x)\right]\Big]
\end{aligned}$$

2 Calculated cross sections

To calculate the cross sections, we need to specify which parton distribution functions (PDFs) to use. We will here use the CTEQ6 [2] PDFs, or more specifically the CTEQ6-DIS PDF. Using this PDF, we can integrate the cross sections given in the previous section over x and y to get the total cross section. In doing this, we have allowed Q^2 to take on any value above $Q^2_{\rm min}$. For CTEQ6-DIS, $Q_{\rm min}$ should in principle be 0.2260 GeV, but extrapolations are used below 1.3 GeV and the routines do not behave very well for the lowest values of Q. Hence, we have decided use $Q_{\rm min} = 0.3$ GeV, where the routines are well-behaved. The difference between these choices only appears at the lowest energies (below a few tens of GeV). The CTEQ6 PDFs are calculated in the range $10^{-6} < x < 1$ and for x values below 10^{-6}, extrapolations are needed. Given the value of $Q_{\rm min}$, these extrapolations are only needed for $E_\nu > 10^4 GeV$ or so. Hence, for the energies we are interested in here, these extrapolations are not important. For higher energies, the result depends on the extrapolation used and we here use the one coded into the CTEQ6 PDF routines themselves. For the other physical parameters we have used

$$\begin{aligned}
M_W &= 80.41 \text{ GeV} \\
M_Z &= 91.188 \text{ GeV} \\
G_F &= 1.16639 \times 10^{-5} \text{ GeV}^{-2}
\end{aligned}$$

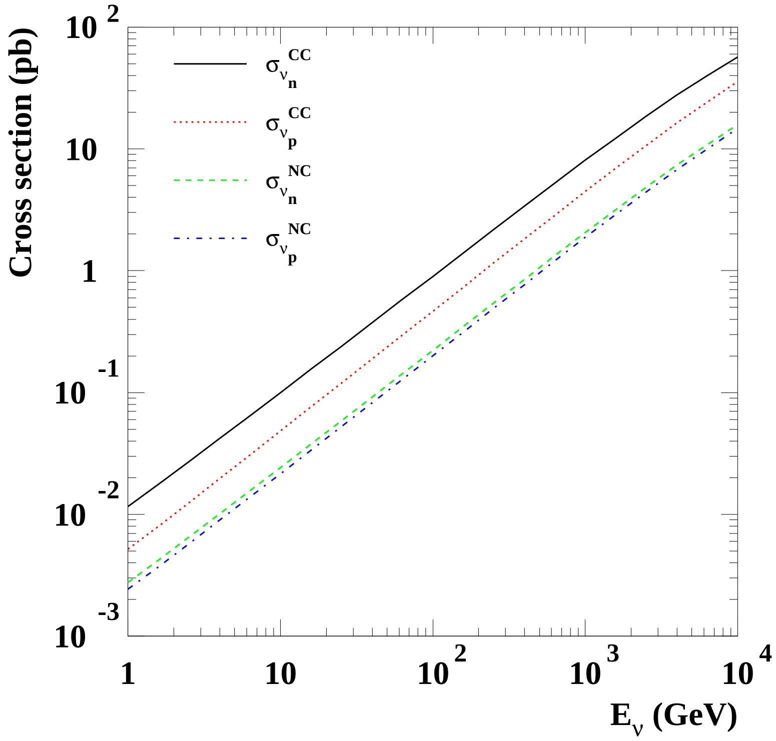

Figure 1: The neutrino nucleon cross section calculated with the CTEQ6-DIS PDF.

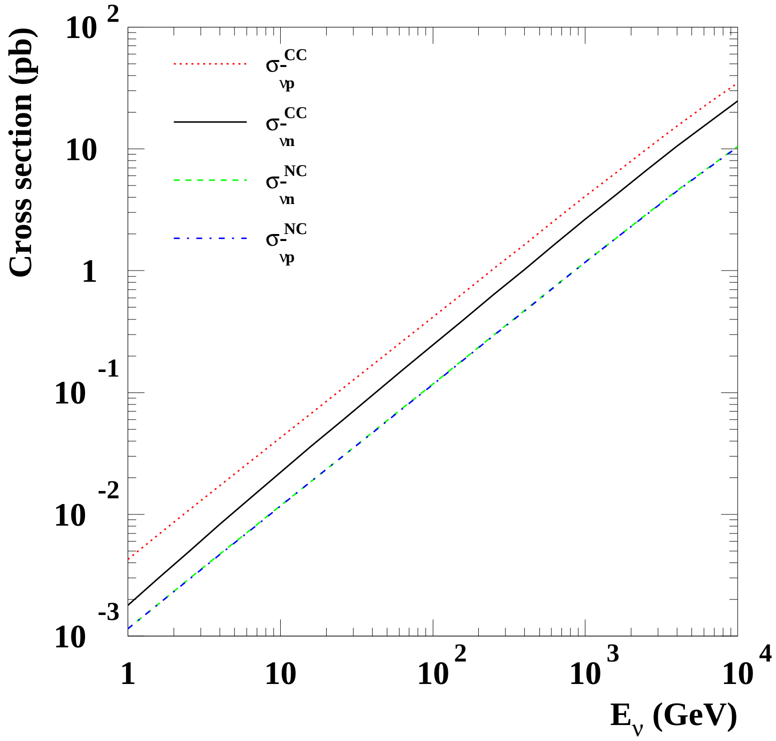

Figure 2: The anti-neutrino nucleon cross section calculated with the CTEQ6-DIS PDF.

Interaction	a [pb]	b
CC $\nu - p$	5.43d-3	0.965
CC $\bar{\nu} - p$	4.59d-3	0.978
CC $\nu - n$	1.23d-2	0.929
CC $\bar{\nu} - n$	2.19d-3	1.022
NC $\nu - p$	2.48d-3	0.953
NC $\bar{\nu} - p$	1.22d-3	0.989
NC $\nu - n$	2.83d-2	0.948
NC $\bar{\nu} - n$	1.23d-3	0.989

Table 1: The fitted parameters of the neutrino-nucleon cross sections in the energy range $10 < E_\nu/\text{GeV} < 10^4$.

$$\begin{aligned} M_p & \quad 0.938272 \text{ GeV} \\ M_n & \quad 0.939566 \text{ GeV} \\ \sin^2\theta_W & = 0.23124 \end{aligned}$$

In Fig. 1, we show the neutrino cross sections on nucleons resulting from these integrations. In Fig. 2, the corresponding cross sections for anti-neutrinos are shown. These cross sections agree well with other calculations in this energy range, e.g. [1] (even though, only cross sections on isoscalar targets are given there). All these total cross sections are tabulated in the energy range $1 < E_\nu/\text{GeV} < 10^{12}$. These tables are then interpolated by a `Fortran` routine `nusigint`.

2.1 Parameterizations

Even though one should use the full integrated cross sections, it is sometimes convenient to use parameterizations. We have made fits of the cross sections in the range $10 < E_\nu/\text{GeV} < 10^4$. Even though the cross sections are approximately linear in this range, we receive a significantly better fit with a function form of the following kind,

$$\sigma = a \left(\frac{E_\nu}{GeV} \right)^b . \tag{1}$$

In Table 1, the fitted parameters are shown. In the fitted range, these parameterizations are good to within roughly 10%. Below 10 GeV and above 10^4 GeV, they overestimate the cross section. The resulted fits are shown in Figs. 3–4.

2.2 Interpolated results

The results of the cross section calculations are also available as a function, `nusigint` that performs interpolations of tables of calculated cross sections. Using this routine is

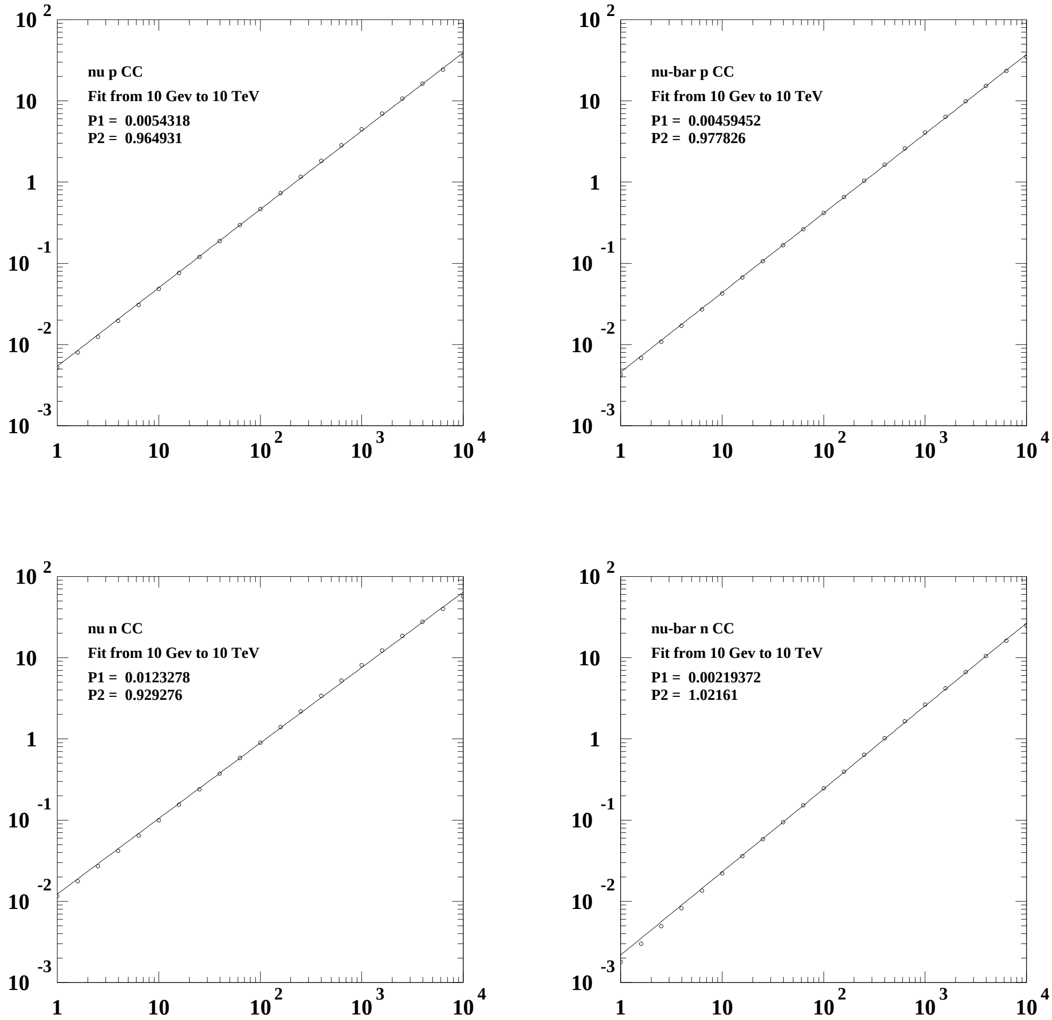

Figure 3: The (anti)neutrino nucleon charged current cross sections and the fits in the range $10 < E_\nu/\mathrm{GeV} < 10^4$.

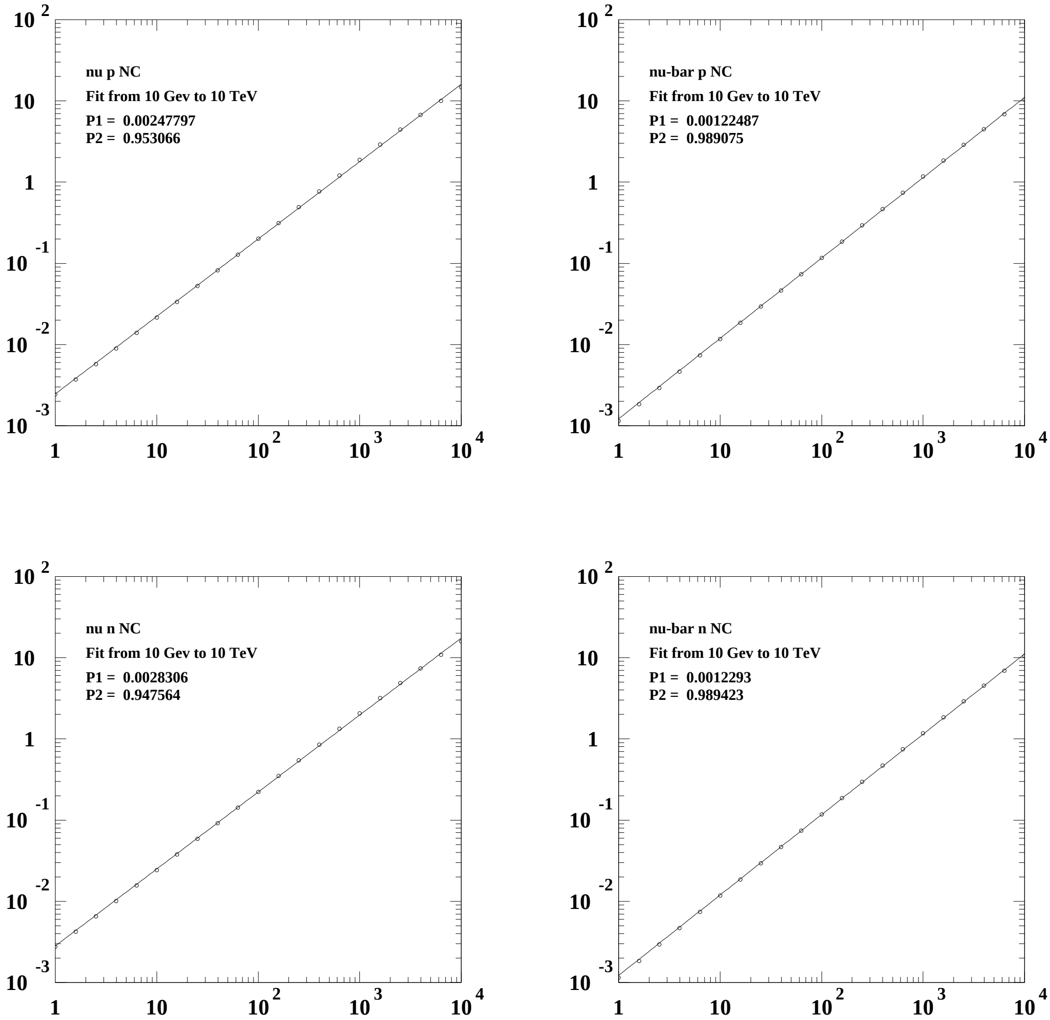

Figure 4: The (anti)neutrino nucleon neutral current cross sections and the fits in the range $10 < E_\nu/\text{GeV} < 10^4$.

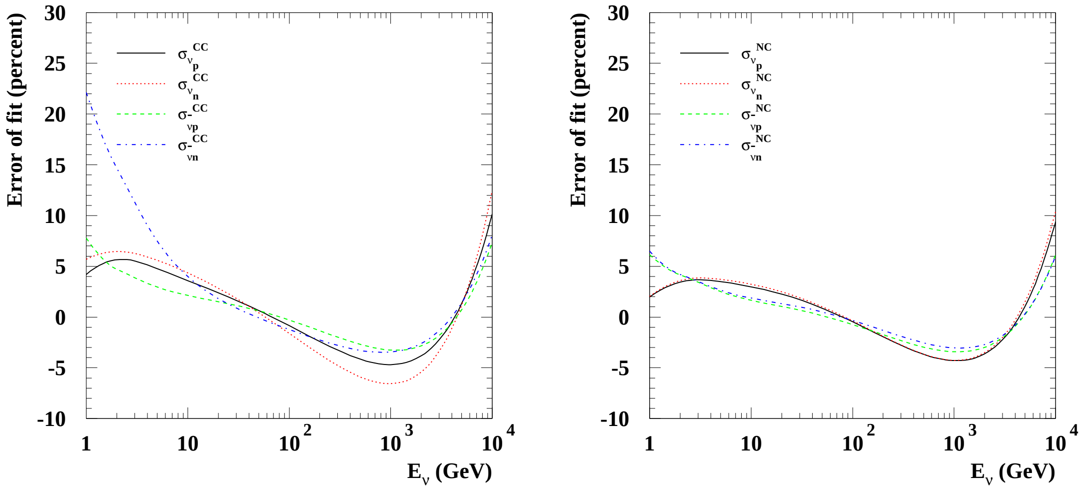

Figure 5: The error of the parameterizations (in %) of the cross sections. The energies are given in GeV.

very fast and the interpolation errors are below 1%. The interpolations work between 1 GeV and 10^{12} GeV.

2.3 Comparison with experiments

The neutrino-nucleon charged current cross sections have been measured by several experiments (see [3]) for a summary. The average cross section in the energy range up to $E_\nu = 350$ GeV, the average is [3]

$$\frac{\sigma_{\nu N}}{E_\nu} = 0.677 \pm 0.014 \times 10^{-38}\ \mathrm{cm}^2/\mathrm{GeV}$$

$$\frac{\sigma_{\bar{\nu} N}}{E_\nu} = 0.334 \pm 0.008 \times 10^{-38}\ \mathrm{cm}^2/\mathrm{GeV}$$

As a comparison, our calculated cross sections at 100 GeV are

$$\frac{\sigma_{\nu N}}{E_\nu = 100\ \mathrm{GeV}} = 0.684 \times 10^{-38}\ \mathrm{cm}^2/\mathrm{GeV}$$

$$\frac{\sigma_{\bar{\nu} N}}{E_\nu = 100\ \mathrm{GeV}} = 0.330 \times 10^{-38}\ \mathrm{cm}^2/\mathrm{GeV}$$

i.e. very close to the measured cross sections. The measured cross sections are shown in Fig. 6.

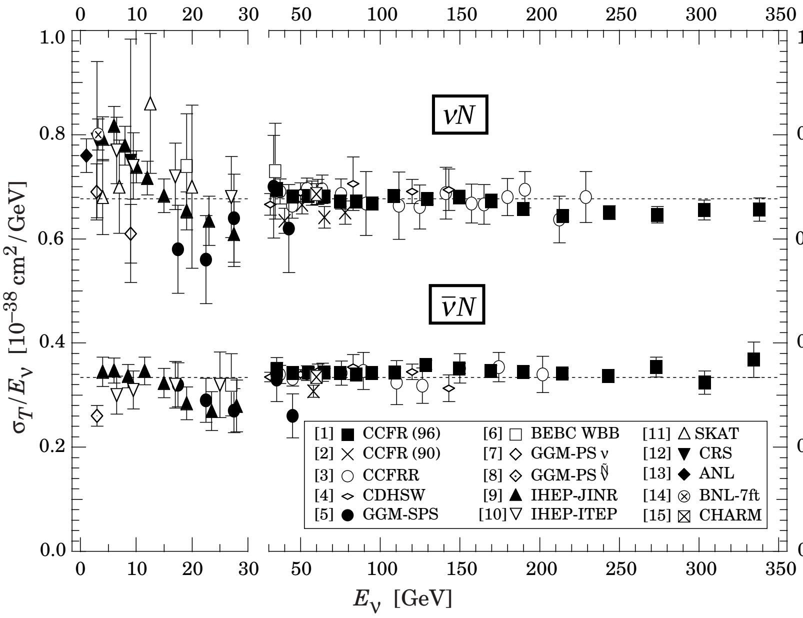

Figure 6: The measured neutrino-nucleon cross sections up to 350 GeV. The horizontal dashed lines are the averages mentioned in the text.

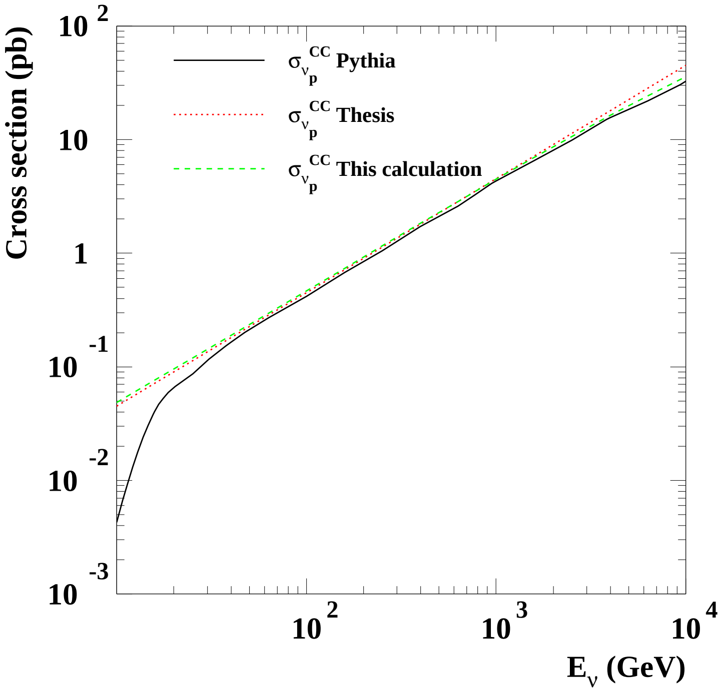

Figure 7: Comparison with cross sections calculated by Pythia. The solid balck curve shows the cross section calculated with the CTEQ5M1 PDF in Pythia, the red dotted shows the cross sections parameterization in [4] and the green dashed shows the cross section calculated with our new code and the CTEQ6-DIS PDF. All cross sections are for charged current neutrino on proton interactions (cross section measured in pb).

2.4 Comparison with Pythia

Pythia can also be used to simulate neutrino nucleon interactions. The latest CTEQ PDFs are not available in Pythia, so for comparison we use the CTEQ5M in Pythia (or rather a parameterization of it, CTEQ5M1). We have set all kinematical cut-offs to as low values as possible in Pythia, not to miss any parts of the parameter space.

In Fig. 7, the cross sections with Pythia and our new calculation can be seen. As can be seen, Pythia shows slightly smaller cross sections, especially at low neutrino energies. This indicates that Pythia has some cut-offs built-in, that cannot be overriden by setting the cut-offs to lower values manually.

To further investigate what could be going on, in Fig 8 we show the distribution of

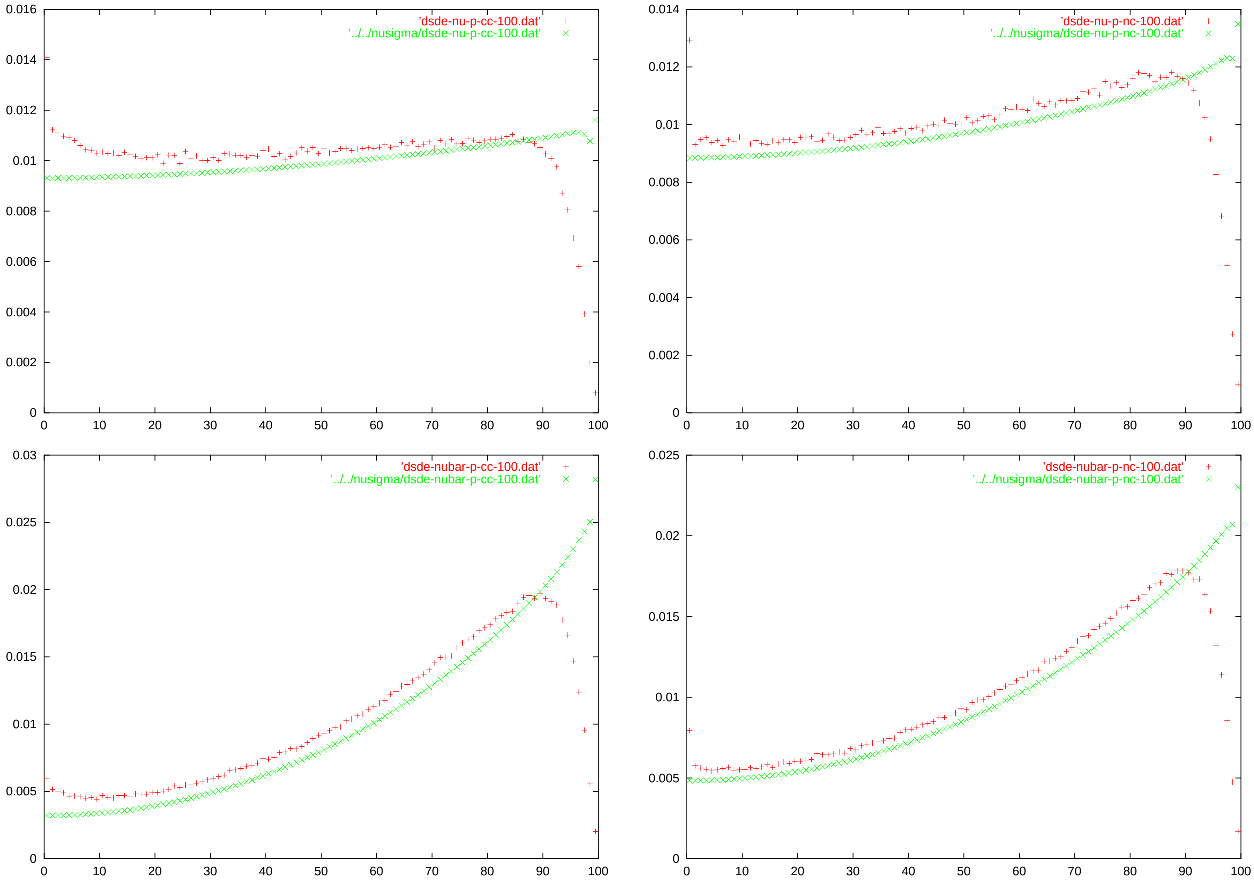

Figure 8: Comparison of final lepton energies with Pythia (red solid) and our new calculation (green dashed). The neutrino energy was 100 GeV in this example. Both curves are normalized to the total cross section, but the Pythia curve also contains secondary leptons showing up as a bump at low energies.

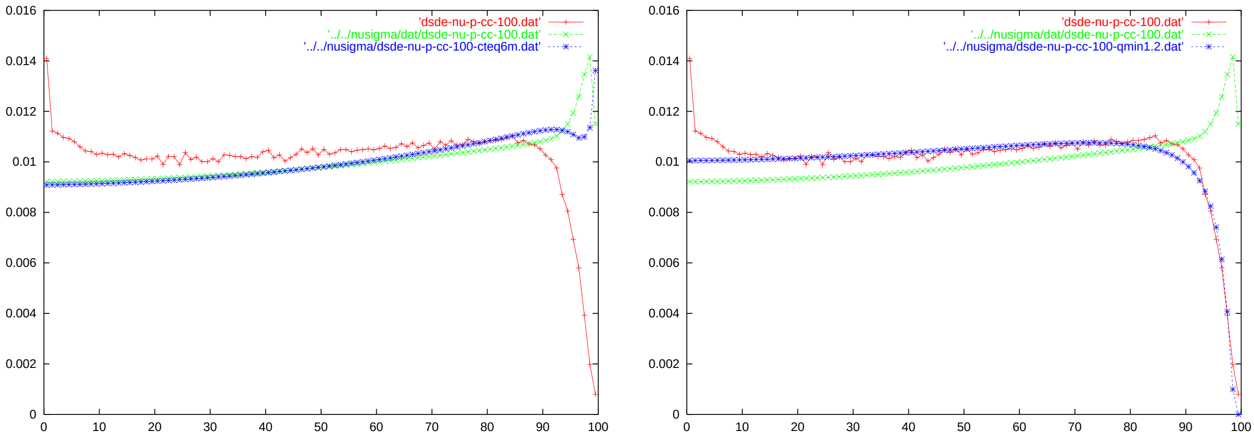

Figure 9: Comparison of final lepton energies with Pythia (solid red) and our new calculation. The left panel shows a comparison if we calculate with CTEQ6M (blue short-dashed) instead of CTEQ6-DIS (green dashed), whereas the right panel shows the difference if we set $Q_{\rm min} = 1.2$ GeV (blue short-dashed) instead of $Q_{\rm min} = 0.2260$ GeV (green dashed). Note that in this figure the green dashed curve (and the blue short-dashed in the left panel) has been calculated with $Q_{\rm min} = 0.2260$ GeV instead of 0.3 GeV that has been used otherwise.

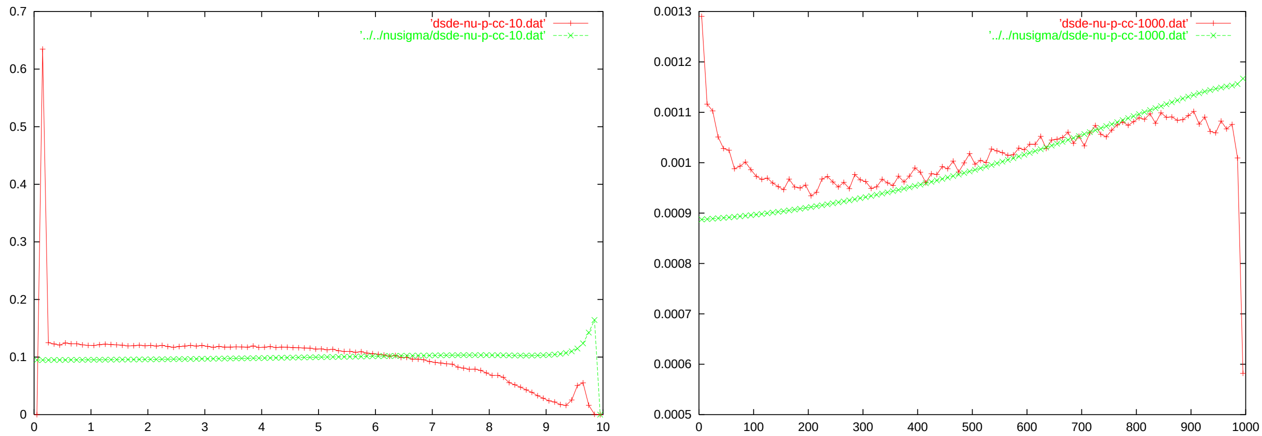

Figure 10: Comparison of final lepton energies with Pythia (red solid) and our new calculation (green dashed). The neutrino energy was 10 GeV in the left panel and 1000 GeV in the right panel. Both curves are normalized to the total cross section, but the Pythia curve also contains secondary leptons showing up as a bump at low energies.

final lepton energies, as obtained with Pythia and directly from our new cross section calculation. Both curves are normalized to the total cross section, but the Pythia curve also contains secondary leptons showing up as a bump at low energies. It is interesting to note that the cross sections differ at the highest energies. To understand this difference, in Fig. 9 we show our calculation compared to Pythia again, but in the left panel we have also used CTEQ6M instead of CTEQ6-DIS. CTEQ6M should be very similar to the PDF used in Pythia, but we don not reproduce Pythia with this one either. The differences seen at high energies indicate that the calculations are uncertain at these high energies. In the right panel of Fig. 9 we set $Q_{\rm min}$ to 1.2 GeV. We can here clearly see that we reproduce the Pythia curve rather well (except at low energies where Pythia has a bump from secondary leptons). Hence we conclude that Pythia's discrepancy seems to come from differences in the low-Q behaviour. Note that the PDFs are not tabulated below $Q = 1.3$, and instead extrapolations are used for these low Q values. It is thus fair to say that it is hard to judge which calculation that should be trusted the most. As mentioned earlier, we choose to fix $Q_{\rm min} = 0.3$ GeV, down to which the CTEQ6-DIS routines are well-behaved.

For comparison, we also show the final lepton energies at 10 GeV and 1000 GeV in Fig. 10. As can be seen, there are differences at these energies as well, especially at low energies.

3 New neutrino Monte Carlo

In the previous section we saw that there are some differences between Pythia and our new calculation of the cross sections. It seems that Pythia has some hard-coded restrictions of the available phase space that could affect our results. Hence, we will use these new calculations to design a new neutrino-nucleon scattering Monte Carlo, `nuNevent`.

3.1 Design of the Monte Carlo

The Monte Carlo consists of one subroutine `nuNevent` which returns on neutrino-nucleon scattering event. As input, it requires the neutrino energy, the neutrino type, the type of target and the interaction type. As output, the neutrino energy and angle as well as the hadronic energy and angle are returned. The MonteCarlo uses the differential cross section, differential in $\ln(x)$ and $\ln(y)$,

$$d\sigma = \frac{d^2\sigma}{d\ln(x)d\ln(y)}$$

and selects a set of points $lx = \ln(x)$ and $ly = \ln(y)$. It then calculates the differential cross section, $d\sigma$ at that given point and if $r < d\sigma/d\sigma^{\rm max}$ that event is kept, otherwise a new set of points is chosen. r is a random number, uniformly distributed between 0 and 1 and $d\sigma^{\max}$ is the maximum of the differential cross section (at that given energy). $d\sigma^{\max}$ has to be determined at any given energy for all the possible interaction types. To gain speed, we determine this maximum beforehand with the program `nusigmaxfind.f`. This program calculates the maximum of the differential cross section at energies from 1 GeV to 10^{12} GeV and stores the result in data files. A routine `nudsdlxdlymax` is then used to read these tables and interpolate them to return the maximum at any given energy for any ineraction type.

The routine `nuNevent` is the routine that generates an event and returns the kinematical variables for the event. The final lepton energy is given by (neglecting lepton masses)

$$E_l = E_\nu(1-y)$$

The angle, θ_l (with respect to the incoming neutrino direction) is given by

$$\sin^2\left(\frac{\theta_l}{2}\right) = \frac{Mxy}{2E_l}$$

where M is the nucleon mass. Using four-momentum conservation, we can also calculate the energy and angle of the remaining hadronic system (neglecting the nucleon remnant),

$$\begin{aligned} E_h &= E_\nu - E_f \\ \sin\theta_h &= -\frac{E_l}{E_h}\sin\theta_l. \end{aligned}$$

These expressions neglect the mass of the struck quark and considers the hadronic jet as a relativistic jet with four-momentum

$$p_h = (E_h, E_h \sin\theta_h, 0, \cos\theta_h).$$

The Monte Carlo scans over the $\ln(x)$ and $\ln(y)$ parameter space within the following ranges

$$\begin{aligned} x &\in [10^{-6} : 1] \\ y &\in [10^{-4} : 1] \end{aligned}$$

which should be sufficient for neutrino energies up to 10 TeV. For higher energies (where lower x values are possible), it might be necessary to decrease the lower end of the x range. `nuNevent` will print a warning message if such a change is needed to cover the full parameter space. `nuNevent` will also warn if a higher maximum of the cross section is found, than that returned from `nudsdlxdlymax` (multiplied by a fudge factor of 1.1 to allow for a small error in the maximum value found).

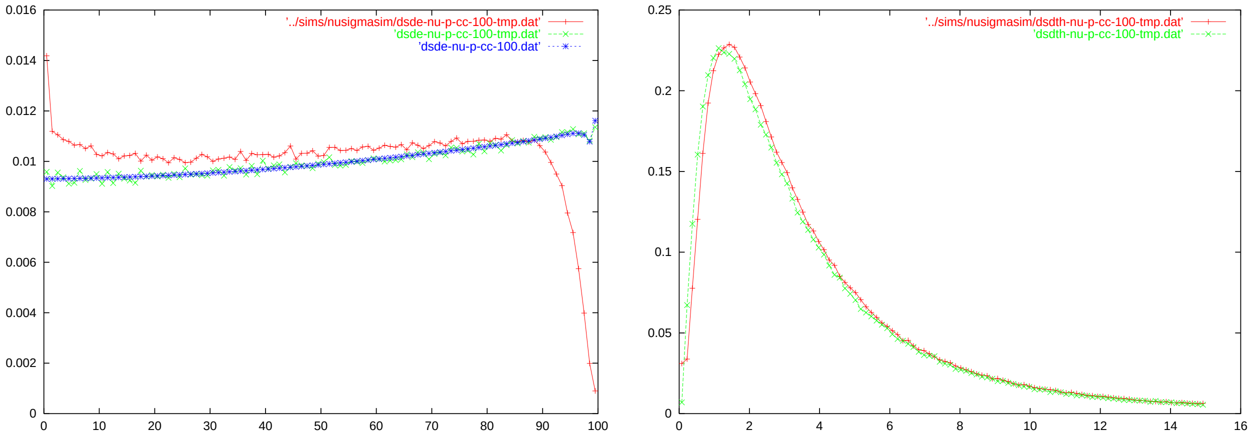

Figure 11: Comparison of the new Monte Carlo and Pythia. In the left panel, the distribution of energies for 1 000 000 events is shown. The red curve is the result with Pythia, the green curve is the result with the new Monte Carlo **nuNevent** and the blue curve is the distribution obtained by integrating the cross section directly. In the right panel, the distribution of lepton angles are shown. Both figures are for charged current neutrino–proton scattering at 100 GeV.

3.2 Comparison with Pythia

To compare the new Monte Carlo with Pythia, we have generated 1 000 000 neutrino-proton charged current events (at 100 GeV) with both programs. The resulting lepton energies and angles are shown in Fig. 11. As can be seen, the new Monte Carlo generates the same distribution as a direct integration of the cross section shows. As seen before, the new routines give more events at higher energies, whereas Pythia also contains a small fraction of low energy secondary leptons. The distribution of angles is also very similar, with slightly smaller angles with the new Monte Carlo (as expected since the energies are on average slightly higher). The generation of these 1 000 000 events took 654 seconds for Pythia and 165 seconds[1] for `nuNevent`, i.e the new Monte Carlo is almost four times faster than Pythia[2].

Hence, the new Monte Carlo seems to work as it should, and it does it fast.

[1]The given CPU seconds are on an Apple Powerbook G4, 1.5GHz.

[2]To be fair it performs more things than `nuNevent`, but those things are not needed here.

References

[1] R. Gandhi, C. Quigg, M.H. Reno and I. Sarcevic, *Neutrino Interactions at Ultrahigh Energies*, Phys. Rev. **D58**(1998)093009. [arXiv: hep-ph/9807264]

[2] J. Pumplin et al., *New generation of parton distributions with uncertainties from global QCD analysis*, JHEP **07**(2002)012. [arXiv: hep-ph/0201195] ; `http://user.pa.msu.edu/wkt/cteq/cteq6/cteq6pdf.html`

[3] Particle Data Group, `http://pdg.lbl.gov/`.

[4] J. Edsjö, PhD thesis.

[5] J.-M. Lévy, *Cross-section and polarization of neutrino-produced* τ*'s made simple*, [arXiv:hep-ph/0407371].